B.Sc. Physics Part-II,

Paper-IV, Group-A

Laplace equation in polar coordinates:—

Laplace equation $\nabla^2 V = 0$ —— (1)

In terms of polar coordinates (ρ, ϕ), we write ∇^2.

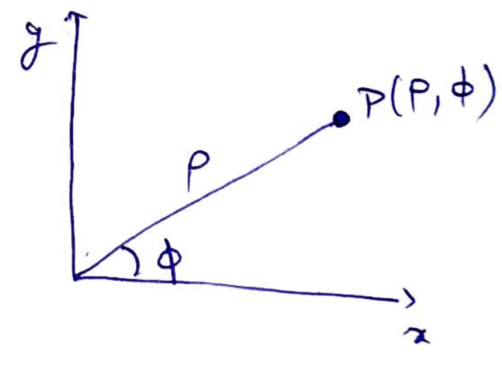
In Cartesian coordinates (2D)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In polar coordinates

$x = \rho \cos\phi$

$y = \rho \sin\phi$



Laplacian in orthogonal curvilinear coordinates is given by

$$\nabla^2 \tilde{\phi} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \tilde{\phi}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \tilde{\phi}}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \tilde{\phi}}{\partial u_3}\right)\right]$$ —— (2)

$h_1, h_2, h_3 \rightarrow$ Scale factors

$u_1, u_2, u_3 \rightarrow$ Curvilinear coordinates

for Palane polar coordinate $(u_1, u_2) \equiv (\rho, \phi)$

Position vector $\vec{r} = x\hat{i} + y\hat{j}$

$$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j}$$ —— (3)

$$\vec{r} = \vec{r}(u_1, u_2) \equiv \vec{r}(\rho, \phi) \quad \text{--- (4)}$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|$$

From (3) & (4)

~~$h_1 = |\cos\phi\,\hat{i} + \sin\phi\,\hat{j}$~~

$$h_\rho = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$h_\phi = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{(\rho\sin\phi)^2 + (\rho\cos\phi)^2} = \rho$$

For 2D

$$\nabla^2\tilde{\phi} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2}{h_1}\frac{\partial \tilde{\phi}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1}{h_2}\frac{\partial \tilde{\phi}}{\partial u_2}\right)\right]$$

$$\text{or } \nabla^2\tilde{\phi} = \frac{1}{h_\rho h_\phi}\left[\frac{\partial}{\partial \rho}\left(\frac{h_\phi}{h_\rho}\frac{\partial \tilde{\phi}}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{h_\rho}{h_\phi}\frac{\partial \tilde{\phi}}{\partial \phi}\right)\right]$$

$$\nabla^2\tilde{\phi} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \tilde{\phi}}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \tilde{\phi}}{\partial \phi}\right)\right]$$

$$\nabla^2\tilde{\phi} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \tilde{\phi}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \tilde{\phi}}{\partial \phi^2} \quad \text{--- (5)}$$

~~$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \rho}\right)$~~

[Note – for details see Part III Mathematical Physics notes uploaded on the website]

Eqn (1) in Plane polar coordinates is written as

$$\nabla^2 V = 0$$

$$\text{or } \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial V}{\partial \rho}\right] + \frac{1}{\rho^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = 0 \quad \text{--- (6)}$$

We use separation of variables ~~to~~ for eq. (6) to obtain solutions:

$$V(\rho,\phi) = R(\rho)\,\Psi(\phi) \quad \text{--- (7)}$$

Now using Eq. (7) in (6), we obtain

$$\Psi(\phi)\,\rho\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial R(\rho)}{\partial\rho}\right) + R(\rho)\,\frac{1}{\rho^2}\,\frac{\partial^2\Psi(\phi)}{\partial\phi^2} = 0$$

multiplying by $\dfrac{\rho^2}{R(\rho)\Psi(\phi)}$,

$$\frac{\rho}{R(\rho)}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{\Psi(\phi)}\,\frac{\partial^2\Psi(\phi)}{\partial\phi^2} = 0$$

or $$\frac{\rho}{R(\rho)}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{R(\rho)}{\partial\rho}\right) = -\frac{1}{\Psi(\phi)}\,\frac{\partial^2\Psi(\phi)}{\partial\phi^2} \quad \text{--- (8)}$$

Since the two terms are separately functions of ρ and ϕ respectively. Therefore, each terms must be constant.

$$\frac{\rho}{R}\,\frac{d}{d\rho}\left(\rho\,\frac{dR}{d\rho}\right) = \nu^2, \quad \frac{1}{\Psi}\,\frac{d^2\Psi}{d\phi^2} = -\nu^2 \quad \text{--- (9)}$$

Now the general solution for $\nu \neq 0$ is

$$R(\rho) = a_\nu \rho^{\nu} + b_\nu \rho^{-\nu} \quad \text{and} \quad \Psi(\phi) = A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}$$

Thus $$V(\rho,\phi) = R(\rho)\,\Psi(\phi)$$

or $$V(\rho,\phi) = \sum_{\nu} \left(a_\nu \rho^{\nu} + b_\nu \rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}\right)$$

For the special case of $\nu = 0$, the solutions are

$$R(\rho) = a_0 + b_0 \ln \rho$$
$$\text{and } \Phi(\phi) = A_0 + B_0 \phi$$

Since from Eqn (9)

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = 0 \quad \text{and} \quad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

The most general solution of Eqn (6) is given by

$$\boxed{V(\rho, \phi) = (a_0 + b_0 \ln \rho)(A_0 + B_0 \phi) + \sum_{\nu \neq 0} (a_\nu \rho^{\nu} + b_\nu \rho^{-\nu})(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi})}$$